Selfduality for coupled Potts models on the triangular lattice

Jean-François Richard[*,†], Jesper Lykke Jacobsen[*], and Marco Picco[†]
[*] *LPTMS, Université Paris-Sud, Bâtiment 100, 91405 Orsay, France and*
[†] *LPTHE, Université Paris VI, Tour 16, 4 place Jussieu, 75252 Paris cedex 05, France*
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We present selfdual manifolds for coupled Potts models on the triangular lattice. We exploit two different techniques: duality followed by decimation, and mapping to a related loop model. The latter technique is found to be superior, and it allows to include three-spin couplings. Starting from three coupled models, such couplings are necessary for generating selfdual solutions. A numerical study of the case of two coupled models leads to the identification of novel critical points.

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I. INTRODUCTION

The two-dimensional Potts model is a well-studied model of statistical mechanics [1] and continues to attract the interest of many workers. Its definition is simple. Given a lattice with vertices $\{i\}$ and edges $\langle ij \rangle$, the Hamiltonian reads

$$\beta H = -K \sum_{\langle ij \rangle} \delta(S_i, S_j), \tag{1}$$

where δ is the Kronecker delta. The spins $S_i = 1, 2, \ldots, q$ initially take q discrete values. However, by making a random cluster expansion [2] it is easily seen that the partition function can be written

$$Z = \sum_{\mathcal{C}} b^{e(\mathcal{C})} q^{n(\mathcal{C})}, \tag{2}$$

where $b = \mathrm{e}^K - 1$. Here, the sum is over the $2^{|\langle ij \rangle|}$ possible colourings $\mathcal{C}$ of the edges (each edge being either coloured or uncoloured), $e(\mathcal{C})$ is the number of coloured edges, and $n(\mathcal{C})$ is the number of connected components (clusters) formed by the coloured edges. Taking Eq. (2) as the *definition* of the Potts model, it is clear that q can now be considered as a real variable, independently of the original spin Hamiltonian. Also, we shall adopt the point of view that Eq. (2) makes sense for any real b, although $b < -1$ would correspond to an unphysical (complex) value of the spin coupling K.

Exact evaluations of Eq. (2), in the sense of the Bethe Ansatz, exist for several lattices and for specific curves in (q, b) space along which the model happens to be integrable [3]. This is true, in particular, for the square lattice with [3, 4]

$$b = \pm\sqrt{q}, \tag{3}$$
$$b = -2 \pm \sqrt{4 - q}, \tag{4}$$

and for the triangular lattice with [5]

$$b^3 + 3b^2 = q. \tag{5}$$

These curves have several features in common. First, they correspond to critical points (with correlation functions decaying as power laws) for $0 \leq q \leq 4$ [6], whose nature can be classified using conformal field theory (CFT) [7]. Second, the values of the coupling constants are often so that the partition function is selfdual (see below); this is the case for the curves (3) and (5) above, whereas the two curves in (4) are mutually dual.

The part of the curves having $b > 0$ corresponds to the ferromagnetic phase transition, whose critical behaviour is lattice independent (universal). More interestingly, the antiferromagnetic ($-1 \leq b < 0$) and unphysical regimes ($b < -1$) contain non-generic critical points whose relation to CFT has, at least in some cases, not been fully elucidated. This is so in particular for $b = -1$, where the Potts model reduces to a colouring problem, and Eq. (2) becomes the chromatic polynomial.

Much less is known about several Potts models, coupled through their energy density $\delta(S_i, S_j)$. Results coming from integrability seem to be limited to the case of $N = 2$ coupled models [8], which on the square lattice only leads to new critical points in the well-studied Ashkin-Teller case [3] (i.e., with $q = 2$). Apart from that, CFT-related

results are essentially confined to perturbative expansions in $\epsilon \sim q-2$ around the ferromagnetic critical point [9, 10]. These results, corroborated by numerical evidence [11, 12], indicate the existence of novel critical points for $N \geq 3$, with possible implications for the random-bond Potts model through the formal analytical continuation (replica limit) $N \to 0$.

In the present publication we investigate the possibility of novel critical behaviour in N coupled Potts models on the triangular lattice. To identify candidate critical points we first search for selfdual theories. In comparison with a similar study on the square lattice [11, 12] several distinctive features emerge due to the non-selfdualness of the lattice. This leads us to use two different techniques. In the first, a standard duality transformation is followed by decimation (star-triangle transformation). This turns out to be quite cumbersome, already for $N = 2$. We therefore turn to a second technique, which utilises a mapping to a system of coupled loop models. This leads to simpler relations, and as a bonus allows to include three-spin couplings around one half of the lattice faces. Starting from $N = 3$ coupled models, such additional couplings are actually necessary for generating non-trivial selfdual solutions.

For $N = 2$ we numerically investigate the non-trivial selfdual manifold. Following the motivation given above, the main interest here is to establish whether a given selfdual point corresponds to a renormalisation group fixed point (and possibly even to a critical fixed point). We shall see that these expectations are indeed born out: the numerics is compatible with critical points whenever $0 \leq q \leq 4$. Measuring the central charge, we identify the corresponding universality classes. These can in some cases be understood from those of a single model, but we also identify points possessing novel critical behaviour.

The paper is laid out as follows. In Section II we present the technique of duality followed by decimation for two coupled models with pure two-spin interactions. In particular, we find a non-trivial selfdual solution. The mapping to a loop model, given in Section III, allows to rederive this solution in a much simpler way, and to generalise to the case where three-spin interactions are included. In Section IV we use this technique to treat the case of three coupled models with both two and three-spin interactions. A numerical study of the non-trivial selfdual solution found in Section II is the object of Section V. Finally, Section VI is devoted to our conclusions.

II. MODELS WITH TWO-SPIN INTERACTIONS

To illustrate the first technique (duality and decimation), we consider the case of $N = 2$ coupled models with two-spin interactions. In order to simplify the notation, we introduce the symbol $\delta_{ij}^{\mu} = \delta\left(S_i^{\mu}, S_j^{\mu}\right)$, where the superscript refers to the spins of the μ'th model ($\mu = 1, 2, \ldots, N$). We are interested in the coupled model defined by the Hamiltonian

$$\beta H_2 = -\sum_{\langle ij \rangle} \left\{ K_1 \delta_{ij}^1 + K_2 \delta_{ij}^2 + K_{12} \delta_{ij}^1 \delta_{ij}^2 \right\}. \tag{6}$$

The spins S_i^{μ} take q_{μ} different values.

A. Duality followed by decimation

As shown in Ref. [12, 13], Eq. (6) admits a (generalised) random cluster expansion resulting in

$$Z = \sum_{\mathcal{C}_1, \mathcal{C}_2} b_1^{e(\mathcal{C}_1 \cap \overline{\mathcal{C}_2})} b_2^{e(\overline{\mathcal{C}_1} \cap \mathcal{C}_2)} b_{12}^{e(\mathcal{C}_1 \cap \mathcal{C}_2)} q_1^{n(\mathcal{C}_1)} q_2^{n(\mathcal{C}_2)}, \tag{7}$$

where $\mathcal{C}_{\mu}$ are independent colourings of the μ'th model, and we have defined the complementary colouring $\overline{\mathcal{C}_{\mu}} \equiv \langle ij \rangle - \mathcal{C}_{\mu}$. The new parameters b are related to the coupling constants K through

$$b_{\mu} = \mathrm{e}^{K_{\mu}} - 1, \qquad b_{12} = \mathrm{e}^{K_1 + K_2 + K_{12}} - \mathrm{e}^{K_1} - \mathrm{e}^{K_2} + 1 \tag{8}$$

As explained in the Introduction, we shall take the point of view that the model is *defined* by Eq. (7) for any real values of b and q_{μ}.

Up to an irrelevant constant, the partition function of the dual model is again given by (7), but now with respect to the dual (hexagonal) lattice, and with dual values $\tilde{b}$ of the parameters [12, 13]:

$$\widetilde{b_1} = \frac{b_2\, q_1}{b_{12}}, \qquad \widetilde{b_2} = \frac{b_1\, q_2}{b_{12}}, \qquad \widetilde{b_{12}} = \frac{q_1\, q_2}{b_{12}}. \tag{9}$$

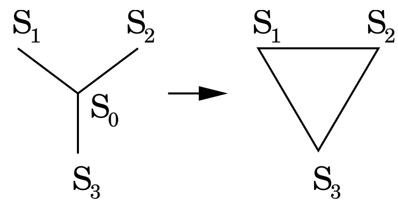

FIG. 1: The star-triangle transformation.

A rather obvious procedure would be to follow (9) by a standard decimation prescription (star-triangle transformation) in order to get back to parameters b' defined with respect to the triangular lattice, and then search for selfdual solutions, $b = b'$. A key assumption, of course, is that such solutions exist within the original parameter space, i.e., with only nearest-neighbour couplings among the spins [14].

The precise setup is shown in Fig. 1. We form the partial trace over all spins S_0^μ situated at even (Y-shaped) vertices of the hexagonal lattice, while keeping the exterior spins S_1^μ, S_2^μ, S_3^μ fixed. Defining $\tilde{b}$ as in Eq. (8), we must have

$$\sum_{S_0^1,S_0^2} \exp\left\{\sum_{i=1}^{3}\left(\widetilde{K_\mu}\sum_{\mu=1}^{2}\delta_{0i}^\mu + \widetilde{K_{12}}\delta_{0i}^1\delta_{0i}^2\right)\right\} = A\exp\left\{\sum_{i>j=1}^{3}\left(K'_\mu\sum_{\mu=1}^{2}\delta_{ij}^\mu + K'_{12}\delta_{ij}^1\delta_{ij}^2\right)\right\}, \tag{10}$$

where the proportionality factor A does not have any bearing on the duality relations for the coupling constants.

We obtain ten distinct relations by considering all symmetry-unrelated choices for the fixed spins S_i^μ with $\mu = 1,2$ and $i = 1,2,3$. Following [14] we suppose $q_\mu \geq 3$ integer initially, and then invoke analytic continuation to claim the validity of the result for arbitrary q_μ.

- For $S_1 \neq S_2 \neq S_3$ on the two lattices:

$$(q_1-3)(q_2-3) + 3(1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}}) + 3(q_2-3)(1+\tilde{b}_1) + 3(q_1-3)(1+\tilde{b}_2) + 6(1+\tilde{b}_1)(1+\tilde{b}_2) = A \tag{11}$$

- For $S_1 = S_2 = S_3$ on the two lattices:

$$(q_1-1)(q_2-1) + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})^3 + (q_2-1)(1+\tilde{b}_1)^3 + (q_1-1)(1+\tilde{b}_2)^3 = A(1+b'_1+b'_2+b'_{12})^3 \tag{12}$$

- For $S_1 = S_2 \neq S_3$ on the two lattices:

$$\begin{aligned}(q_1-2)(q_2-2) + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})^2 + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}}) + (q_2-2)(1+\tilde{b}_1)^2 + (q_1-2)(1+\tilde{b}_2)^2 +\\ (q_2-2)(1+\tilde{b}_1) + (q_1-2)(1+\tilde{b}_2)(1+\tilde{b}_1)^2(1+\tilde{b}_2) + (1+\tilde{b}_2)^2(1+\tilde{b}_1) = A(1+b'_1+b'_2+b'_{12})\end{aligned} \tag{13}$$

- For $S_1^1 = S_2^1 \neq S_3^1$ and $S_1^2 \neq S_2^2 = S_3^2$:

$$\begin{aligned}(q_1-2)(q_2-2) + (q_2-2)(1+\tilde{b}_1)^2 + (q_2-2)(1+\tilde{b}_1) + (q_1-2)(1+\tilde{b}_2)^2 + (q_1-2)(1+\tilde{b}_2) +\\ (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_1)(1+\tilde{b}_2) + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_1) + (1+\tilde{b}_1)(1+\tilde{b}_2) +\\ (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_2) = A(1+b'_1)(1+b'_2)\end{aligned} \tag{14}$$

- For $S_1^1 = S_2^1 = S_3^1$ and $S_1^2 \neq S_2^2 \neq S_3^2$:

$$(q_1-1)(q_2-3) + (q_2-3)(1+\tilde{b}_1)^3 + 3(q_1-1)(1+\tilde{b}_2) + 3(1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_1)^2 = A(1+b'_1)^3 \tag{15}$$

- For $S_1^1 = S_2^1 = S_3^1$ and $S_1^2 = S_2^2 \neq S_3^2$:

$$\begin{aligned}(q_2-2)(q_1-1) + (q_1-1)(1+\tilde{b}_2)^2 + (q_1-1)(1+\tilde{b}_2) + (q_2-2)(1+\tilde{b}_1)^3 +\\ (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})^2(1+\tilde{b}_1) + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_1)^2 = A(1+b'_1+b'_2+b'_{12})(1+b'_1)^2\end{aligned} \tag{16}$$

- For $S_1^1 = S_2^1 \neq S_3^1$ and $S_1^2 \neq S_2^2 \neq S_3^2$:

$$\begin{aligned}(q_1-2)(q_2-3) + (q_2-3)(1+\tilde{b}_1)^2 + (q_2-3)(1+\tilde{b}_1) + (q_1-2)3(1+\tilde{b}_2) +\\ 2(1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}})(1+\tilde{b}_1) + (1+\tilde{b}_1)^2(1+\tilde{b}_2) + (1+\tilde{b}_1+\tilde{b}_2+\widetilde{b_{12}}) + 2(1+\tilde{b}_1)(1+\tilde{b}_2) = A(1+b'_1)\end{aligned} \tag{17}$$

- Eqs. (15), (16) and (17) each represent a pair of relations of which we have only written one representative; the other one is obtained by permuting the two models, i.e., by letting $q_1 \leftrightarrow q_2$ and $b_1 \leftrightarrow b_2$.

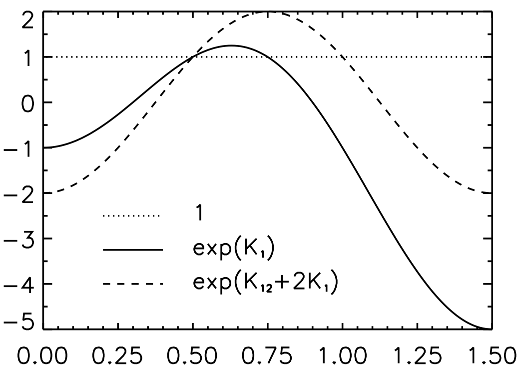

FIG. 2: Variation of the Boltzmann weights (20) with the parameter g defined in Eq. (19).

B. General structure of trivial solutions

The list of selfdual solutions of N coupled models, $\mu = 1, 2, \ldots, N$, will in general contain a certain number of trivial solutions. By a trivial solution we mean one which is a consequence of the selfduality of a single model. Let us discuss in detail two classes [12, 13] of trivial solutions:

1. The models are actually decoupled. This happens, e.g., in the above example with $N = 2$ when $K_{12} = 0$, that is $b_{12} = b_1 b_2$. We will then have $b_\mu^3 + 3b_\mu^2 = q_\mu$ for each μ, cf. Eq. (5). The number of real solutions of the μ'th equation is $n_\mu = 3$ when $0 < q_\mu < 4$, $n_\mu = 2$ when $q_\mu = 0$ or $q_\mu = 4$, and $n_\mu = 1$ otherwise; there will therefore be $n = \prod_{\mu=1}^N n_\mu$ trivial solutions for the system of N decoupled models.

2. The models couple strongly so as to form a *single* $q = \prod_{\mu=1}^N q_\mu$ state model. This happens when only the coupling constant involving all the models is nonzero. E.g., in the above example with $N = 2$, one would have $K_1 = K_2 = 0$, that is $b_1 = b_2 = 0$. The number of such solutions equals the number of real solutions of Eq. (5), with b replaced by b_{12}.

The goal of our study is to show that there exists selfdual solutions of coupled Potts models on the triangular lattice which are *not* trivial in the above sense.

C. Non-trivial solutions

Let us return to the Hamiltonian (6). We have numerically solved the ten relations (11)–(17) for several different values of q_1 and q_2. The conclusion is that for $q_1 \neq q_2$ only trivial solutions exist.

For $q_1 = q_2$ the situation is different. There are now only seven distinct relations (11)–(17), since the three relations which formerly occurred in pairs will now collapse into single relations. The parameters b are thus less constrained, and accordingly we find non-trivial solutions. Numerically we find that these solutions have $b_1 = b_2$ (but note that there are still trivial solutions with $K_{12} = 0$ which break this symmetry).

Setting now $q \equiv q_1 = q_2$ and $b_1 = b_2$ we can obtain the non-trivial solutions analytically, e.g., by solving Eqs. (11), (13) and (17) for A, b_1 and b_{12}, and verifying that the found solution satisfies all the other relations. The result is:

$$b_1^3 + 6b_1^2 + 3b_1 q + q(q-2) = 0, \qquad b_{12} = \frac{q - b_1^2}{2 + b_1}. \tag{18}$$

For each $q \in (0, 4)$ Eq. (18) admits three distinct solution for b_1. To make clear in the following exactly to which solution we are referring, it is convenient to recast (18) in parametric form, by setting $q = 4\cos^2(\pi g)$. When the parameter g runs through the interval $[0, \frac{3}{2}]$, the number of states q runs through the interval $[0, 4]$ three times. We have then

$$b_1 = x(1-x), \qquad b_{12} = (x-1)^2, \qquad x \equiv 2\cos\left(\frac{2\pi}{3} g\right). \tag{19}$$

This parametrisation also has the advantage over (18) that it is non-singular as $g \to 1$ (i.e., $x \to -1$ and $q \to 4$) and yields the correct limiting values of b_1 and b_{12}.

In terms of x, the Boltzmann weights for two neighbouring spins being identical in none, one, or both of the two models read:

$$1, \qquad \mathrm{e}^{K_1} = 1 + x - x^2, \qquad \mathrm{e}^{K_{12}+2K_1} = 2 - x^2. \tag{20}$$

Their variation with g is shown in Fig. 2. Note that the K_{12} coupling is physical ($\mathrm{e}^{K_{12}} \geq 0$) for $\frac{3}{8} \leq g \leq \frac{9}{8}$, and that K_1 is physical for $\frac{3}{10} \leq g \leq \frac{9}{10}$.

It is also interesting to remark that in the x-parametrisation, Eq. (5) for a single model reads $b = x - 1$; the trivial solution of type 1 is then $b_1 = x - 1$, $b_{12} = (b_1)^2$ with the *same* value of b_{12} as in Eq. (19).

D. Special points on the curve (19)

Let us remark on a few special values of the parameter g for which the physics of the two coupled models can be related to that of a single model.

1. For $g = 1$ (i.e., $q = 4$ and $x = -1$) one has $K_{12} = 0$, whence the models are decoupled.
2. For $g = \frac{3}{4}$ (i.e., $q = 2$ and $x = 0$) one has $K_1 = 0$ and $K_{12} = \log 2$. Thus, the two models couple strongly to form a single $q^2 = 4$ state model at the ferromagnetic fixed point.
3. For $g = \frac{1}{2}$ (i.e., $q = 0$ and $x = 1$) one has $K_1 = K_{12} = 0$. This is an infinite-temperature limit, whose properties depend on the ratio K_1/K_{12} as $g \to \frac{1}{2}$. In fact, for $x \to 1$ we find $K_1 \sim (1-x)$ and $K_{12} \sim 2(1-x)^3$, whence the coupling between the two models is negligible. Note also that $q \sim 3(1-x)^2$, whence the point $(q, b_1) = (0, 0)$ is approached with infinite slope, as is the case for a single Potts model *along* the selfdual curve (5).

Note also that these values of g correspond to degeneracies of the Boltzmann weights (20), cf. Fig. 2.

As to the critical behaviour of a single Potts model, the situation is well understood in the case of the square lattice [7]. There are three critical phases, referred to as ferromagnetic, Berker-Kadanoff and antiferromagnetic in Ref. [7]. By universality, one would expect the same three critical phases to describe the distinct branches of the selfdual curve (5), as is confirmed by numerical transfer matrix results [16]. In particular, for the central charge along (5) one has

$$\begin{aligned}
c &= 1 - \frac{6g^2}{1-g}, && \text{for } 0 \leq g < \frac{1}{2} && \text{(ferromagnetic phase)} \\
c &= 1 - \frac{6g^2}{1-g}, && \text{for } \frac{1}{2} \leq g < 1 && \text{(Berker-Kadanoff phase)} \\
c &= 2 - 6(g-1), && \text{for } 1 < g \leq \frac{3}{2} && \text{(antiferromagnetic phase)}
\end{aligned} \tag{21}$$

when parametrising $q = 4\cos^2(\pi g)$ as in (19). Note that points $(q, b_1) = (0, 0)$ and $(q, b_1) = (4, -2)$ are special, and the critical behaviour when approaching the curve (5) at these points depends on the exact prescription for taking the limit.

We conclude that the above-mentioned special points on the selfdual curve (19) for two coupled models lead to the central charges $c = -4$ for $g = \frac{1}{2}$ and $c = 1$ for $g = \frac{3}{4}$. For $g \to 1$, the result is for the moment uncertain, due to the ambiguity in taking the limit just referred to.

III. TWO AND THREE-SPIN INTERACTIONS

The results of the preceding section can be generalised to a model defined by the Hamiltonian $H = H_2 + H_3$. The term H_2 is as in Eq. (6), whereas H_3 introduces interactions between the three spins around the up-pointing faces $\langle ijk \rangle$ of the triangular lattice, as shown in Fig. 3:

$$\beta H_3 = - \sum_{\langle ijk \rangle} \left\{ L_1 \delta^1_{ijk} + L_2 \delta^2_{ijk} + L_{12} \delta^1_{ijk} \delta^2_{ijk} \right\}. \tag{22}$$

Introducing such interactions around *every* face seems difficult, even in the case of a single model [1].

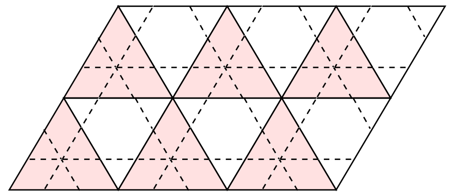

FIG. 3: The lattice of Potts spins is shown in solid linestyle. There are two-spin interactions along every edge, and three-spin interactions among the spins surrounding the up-pointing faces (shaded). The loop model is defined on a shifted triangular lattice, shown in dashed linestyle.

One could of course consider also interactions between two spins in one model and three spins in the other. It can be verified that the method given below can be adapted to this case. However, we have chosen not to consider any further such mixed interactions.

In the following it is convenient to introduce the parameters

$$y_\mu = \mathrm{e}^{L_\mu} - 1, \qquad y_{12} = \mathrm{e}^{L_{12}+L_1+L_2} - \mathrm{e}^{L_1} - \mathrm{e}^{L_2} + 1 \tag{23}$$

in analogy with Eq. (8). Guided by our results without three-spin interactions, we shall assume in the following that non-trivial selfdual solutions only exist when coupling identical models. Thus we restrict the study to $q \equiv q_1 = q_2$, $b_1 = b_2$ and $L_1 = L_2$.

A. Mapping to a fully-packed loop model

Wu and Lin [15] have shown how to produce duality relations for a single Potts model with two- and three-spin interactions, by mapping it to a related loop model. After briefly reviewing their method, we shall adapt it to the case of coupled models.

In Fig. 3 we show the triangular lattice of Potts spins, and the shifted triangular lattice on which the loop model is defined. To obtain the correspondence, one first rewrites the Boltzmann weight around an up-pointing triangle $\langle ijk \rangle$ as

$$w_{ijk} = f_1\delta_{ij} + f_2\delta_{jk} + f_3\delta_{ik} + f_4 + f_5\delta_{ijk} \equiv \sum_{a=1}^{5} f_a \delta_a, \tag{24}$$

where $\delta_1 \equiv \delta_{ij} = \delta(S_i, S_j)$ etc. Note that $f_4 = 1$. To each of the five terms in this sum is associated a link diagram on $\langle ijk \rangle$, as shown in the first line of Fig. 4, indicating which spins participate in the delta symbol. The partition function is then

$$Z_{\rm Potts} = \sum_{S_i} \prod_{\langle ijk \rangle} w_{ijk} = \sum_{\mathcal{L}} q^{n(\mathcal{L})} \prod_{a=1}^{5} f_a^{n_a(\mathcal{L})}, \tag{25}$$

where the sum is over all link diagrams $\mathcal{L}$ for the whole lattice, $n(\mathcal{L})$ is the number of connected components in $\mathcal{L}$, and $n_a(\mathcal{L})$ is the number of up-pointing triangles whose link diagram is of type $a = 1, 2, \ldots, 5$.

The link diagrams are now mapped to fully-packed loop configurations on a shifted triangular lattice (cf. Fig. 3) via the correspondence given in the second line of Fig. 4. The partition function of the loop model is defined as

$$Z_{\rm loop} = \sum_{\mathcal{L}'} z^{p(\mathcal{L}')} \prod_{a=1}^{5} c_a^{n_a(\mathcal{L}')}, \tag{26}$$

where $p(\mathcal{L}')$ is the number of closed polygons (loops) in the loop configuration $\mathcal{L}'$ (which is in one-to-one correspondence with $\mathcal{L}$ using Fig. 4), and z is the fugacity of a polygon. Using now the Euler relation one finds that $Z_{\rm Potts} = Z_{\rm loop}$ if $z = \sqrt{q}$, $c_a = f_a$ for $a = 1, 2, 3$, $c_4 = q^{1/2} f_4$, and $c_5 = q^{-1/2} f_5$ [15].

Finally, the duality relation follows from the invariance of the loop model under $\pi/3$ rotations; notice that this cyclically interchanges the link diagrams of types 1, 2, 3 and permutes the diagrams of type 4, 5.

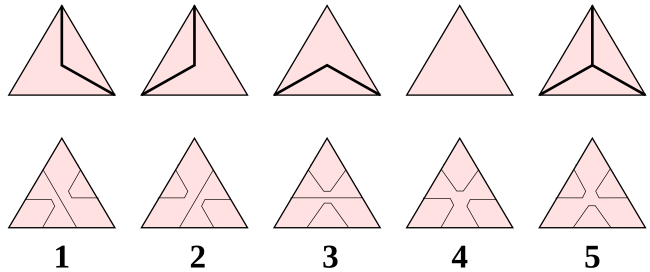

FIG. 4: Correspondence between link diagrams $\mathcal{L}$ for the Potts model (first line) and vertices $\mathcal{L}'$ of the fully-packed loop model (second line).

B. $N = 2$ and mapping to coupled loop models

The above mapping can be adapted to the case of N coupled models. We here consider $N = 2$. The Boltzmann weight w_{ijk} around an up-pointing triangle $\langle ijk \rangle$ can be written in a form generalising (24):

$$w_{ijk} = \sum_{a,b=1}^{5} f_{ab} \delta^1_a \delta^2_b \,; \tag{27}$$

as usual δ^μ refers to the μ'th model. As the two models are identical, we have the symmetry $f_{ab} = f_{ba}$. When $L_1 = L_2 = L_{12} = 0$, the couplings are denoted f^0_{ab}; they are related to the b through

$$\begin{aligned}
f^0_{ii} &= b_{12}, \\
f^0_{ij} &= b_1^2, \\
f^0_{i4} &= b_1, \\
f^0_{i5} &= b_{12}(b_1^2 + 2b_1) + b_1^3, \\
f^0_{44} &= 1, \\
f^0_{45} &= b_1^3 + 3b_1^2, \\
f^0_{55} &= b_{12}^3 + 3b_{12}^2(1 + 2b_1) + 6b_{12}b_1^2,
\end{aligned} \tag{28}$$

where $i \neq j$ are any numbers in $\{1, 2, 3\}$. In the general case ($L \neq 0$) we then have

$$\begin{aligned}
f_{14} &= f^0_{14}, \\
f_{15} &= f^0_{15} + y_1(f^0_{11} + 2f^0_{12} + f^0_{14} + f^0_{15}), \\
f_{44} &= f^0_{44}, \\
f_{55} &= f^0_{55} + 2y_1(3f^0_{15} + f^0_{45} + f^0_{55}) + y_{12}(3f^0_{11} + 6f^0_{12} + 6f^0_{14} + 6f^0_{15} + f^0_{44} + 2f^0_{45} + f^0_{55})\,;
\end{aligned} \tag{29}$$

note that we here only give the f_{ab} needed in the duality relation (33) below.

The Potts model partition function reads

$$Z_{\rm Potts} = \sum_{\mathcal{L}_1, \mathcal{L}_2} q^{n(\mathcal{L}_1) + n(\mathcal{L}_2)} \prod_{a,b=1}^{5} f_{ab}^{n_{ab}(\mathcal{L}_1, \mathcal{L}_2)}. \tag{30}$$

Here, $n_{ab}(\mathcal{L}_1, \mathcal{L}_2)$ is the number of up-triangles where model 1 is in the link state a and model 2 is in the link state b. On the other hand, we can define a coupled loop model through

$$Z_{\rm loop} = \sum_{\mathcal{L}'_1, \mathcal{L}'_2} z^{p(\mathcal{L}'_1) + p(\mathcal{L}'_2)} \prod_{a,b=1}^{5} c_{ab}^{n_{ab}(\mathcal{L}'_1, \mathcal{L}'_2)}. \tag{31}$$

The equivalence between the two goes through as before. Using the Euler relation, one finds $Z_{\rm Potts} = Z_{\rm loop}$ provided that $z = \sqrt{q}$ and that

$$c_{ab} = q^{(\delta_{a,4} + \delta_{b,4} - \delta_{a,5} - \delta_{b,5})/2} f_{ab}. \tag{32}$$

It should now be obvious how the mapping generalises to an arbitrary number of coupled models. One has $c_{ab\cdots} = q^{(N_4-N_5)/2} f_{ab\cdots}$, where N_4 (resp. N_5) counts the number of indices equal to 4 (resp. 5).

The selfduality criterion is again obtained by requiring invariance of $Z_{\rm loop}$ under $\pi/3$ rotations. This means that the c_{ab} are invariant under independent permutations of the indices $\{1,2,3\}$ and of $\{4,5\}$. We also recall the invariance under a permutation of the two models, $c_{ab} = c_{ba}$. Actually, since the two Potts model were taken to be identical and isotropic from the outset, the only non-trivial selfduality criteria are:

$$c_{44} = c_{55}, \qquad c_{14} = c_{15}. \tag{33}$$

C. Selfdual solutions

We now wish to express the condition of selfduality in terms of the parameters b and y, cf. Eqs. (8) and (23).

When three-spin interactions are absent ($L = 0$), the relations (33) immediately yield the solutions given in Sec. II C. This is a remarkable simplification when compared to solving the system of relations (11)–(17); indeed many of these relations turn out to be dependent.

When $L \neq 0$, Eq. (33) still gives the complete solution to the selfduality problem, but it does not generically lead to simple expressions in terms of the parameters b and y. We therefore concentrate on a few remarkable solutions. As before, there are two types of trivial solutions:

1. Trivial decoupled solution ($K_{12} = L_{12} = 0$, or $b_{12} = b_1^2$ and $y_{12} = y_1^2$). One finds the selfduality criterion of a *single* model [5, 15]
$$b_1^3 + 3\,b_1^2 - q + y_1(1+b_1)^3 = 0. \tag{34}$$
2. Trivial strongly coupled solution ($K_1 = L_1 = 0$, or $b_1 = y_1 = 0$). We find
$$b_{12}^3 + 3b_{12}^2 - q^2 + y_{12}(1+b_{12})^3 = 0. \tag{35}$$
This is just the selfduality criterion of a single q^2 state model.

Some noteworthy non-trivial solutions can be found by giving particular values to b_1, y_1 or y_{12}:

1. For $y_1 = 0$ (i.e., $L_1 = 0$), there is only one non-trivial solution:
$$\left(b_1^3 + 6b_1^2 + 3qb_1 + q(q-2)\right)\left(b_1^3 + 3b_1^2 - q\right) = y_{12}(b_1^2 + 5b_1 + q + 2)^3, \qquad b_{12} = \frac{q - b_1^2}{2 + b_1}. \tag{36}$$
Note that when $y_{12} = 0$, the factorisation of the left-hand side allows us to retrieve either (18) or the trivial solution (34).
2. For $y_{12} = y_1^2$ (i.e., $L_{12} = 0$), there is a solution of the form:
$$b_1 = -\frac{q}{2}, \qquad b_{12} = -\frac{q^2}{2} + 3q - 3, \qquad y_1 = \frac{q(4-q)}{(q-2)^2}. \tag{37}$$
3. For $b_1 = -1$ (i.e., $K_1 \to -\infty$), there is a solution of the form:
$$b_{12} = q - 1, \qquad y_1 = -\frac{1}{2}\left(y_{12} + \frac{q^2 - 5q + 5}{q^2 - 4q + 4}\right). \tag{38}$$

IV. THREE COUPLED MODELS

The technique of mapping to coupled loop models has permitted us to study the case of $N = 3$ coupled Potts models, defined by the Hamiltonian

$$\begin{aligned} \beta H \;=\; & -\sum_{\langle ij\rangle}\left\{K_1 \sum_{\mu=1}^{3} \delta_{ij}^{\mu} + K_{12} \sum_{\mu>\nu=1}^{3} \delta_{ij}^{\mu}\delta_{ij}^{\nu} + K_{123}\delta_{ij}^1\delta_{ij}^2\delta_{ij}^3\right\} \\ & -\sum_{\langle ijk\rangle}\left\{L_1 \sum_{\mu=1}^{3} \delta_{ijk}^{\mu} + L_{12} \sum_{\mu>\nu=1}^{3} \delta_{ijk}^{\mu}\delta_{ijk}^{\nu} + L_{123}\delta_{ijk}^1\delta_{ijk}^2\delta_{ijk}^3\right\}. \end{aligned} \tag{39}$$

Since in the case of two coupled models, nontrivial selfdual solutions were only found when coupling identical models in an isotropic way, we shall henceforth restrict the coupling constants as follows:

$$K_1 = K_2 = K_3, \qquad K_{12} = K_{13} = K_{23}, \qquad L_1 = L_2 = L_3, \qquad L_{12} = L_{13} = L_{23}, \tag{40}$$

and we will use the parameters [12, 13]

$$b_1 = \mathrm{e}^{K_1} - 1, \qquad b_{12} = \mathrm{e}^{K_{12}+2K_1} - 2\mathrm{e}^{K_1} + 1, \qquad b_{123} = \mathrm{e}^{K_{123}+3K_{12}+3K_1} - 3\mathrm{e}^{K_{12}+2K_1} + 3\mathrm{e}^{K_1} - 1 \tag{41}$$

$$y_1 = \mathrm{e}^{L_1} - 1, \qquad y_{12} = \mathrm{e}^{L_{12}+2L_1} - 2\mathrm{e}^{L_1} + 1, \qquad y_{123} = \mathrm{e}^{L_{123}+3L_{12}+3L_1} - 3\mathrm{e}^{L_{12}+2L_1} + 3\mathrm{e}^{L_1} - 1. \tag{42}$$

The mapping to coupled loop models follows the obvious generalisation of Eqs. (30)–(31), and the equivalence criterion is stated in the remark after Eq. (32).

To relate the coupling constants f_{abc} to the b and y, we first consider the case of vanishing three-spin interactions (i.e., $y = y_{12} = y_{123} = 0$). Letting $i \neq j \neq k$ designate distinct numbers in $\{1, 2, 3\}$ we have:

$$\begin{aligned}
f^0_{iii} &= b_{123}, \\
f^0_{iij} &= b_1 b_{12}, \\
f^0_{ijk} &= b_1^3, \\
f^0_{ii4} &= b_{12}, \\
f^0_{ii5} &= b_{123} b_1^2 + 2b_{123} b_1 + b_{12} b_1^2, \\
f^0_{ij4} &= b_1^2, \\
f^0_{ij5} &= b_{12}^2 (1 + b_1) + 2b_{12} b_1^2, \\
f^0_{i44} &= b_1, \\
f^0_{i45} &= b_{12}(b_1^2 + 2b_1) + b_1^3, \\
f^0_{i55} &= 2b_{12}^3 + 5b_{12}^2 b_1 + 2b_{123} b_{12} + 4b_{123} b_{12} b_1 + 2b_{123} b_1^2 + b_{123} b_{12}^2, \\
f^0_{444} &= 1, \\
f^0_{445} &= b_1^3 + 3b_1^2, \\
f^0_{455} &= b_{12}^3 + 3b_{12}^2 + 6b_{12} b_1^2 + 6b_{12}^2 b_1, \\
f^0_{555} &= 6b_{12}^3 + 18b_{12}(b_1 + b_{12}) b_{123} + 3(1 + 3b_1 + 3b_{12}) b_{123}^2 + b_{123}^3.
\end{aligned} \tag{43}$$

For the general case one then has (note that we only give those f_{abc} which will be used in the duality relation (45) below):

$$\begin{aligned}
f_{114} &= f^0_{114}, \\
f_{115} &= f^0_{115} + y_1(f^0_{111} + 2f^0_{112} + f^0_{114} + f^0_{115}), \\
f_{124} &= f^0_{124}, \\
f_{125} &= f^0_{125} + y_1(2f^0_{112} + f^0_{123} + f^0_{124} + f^0_{125}), \\
f_{144} &= f^0_{144}, \\
f_{155} &= f^0_{155} + 2y(f^0_{115} + 2f^0_{125} + f^0_{145} + f^0_{155}) + y_{12}(f^0_{111} + f^0_{144} + f^0_{155} + 6f^0_{112} + 2f^0_{123} + \\
&\quad 2f^0_{114} + 2f^0_{115} + 4f^0_{124} + 4f^0_{125} + 2f^0_{145}), \\
f_{444} &= f^0_{444}, \\
f_{445} &= f^0_{445} + y(3f^0_{144} + f^0_{444} + f^0_{445}), \\
f_{455} &= f^0_{455} + 2y(3f^0_{145} + f^0_{445} + f^0_{455}) + y_{12}(3f^0_{114} + f^0_{444} + f^0_{455} + 6f^0_{124} + 6f^0_{144} + 6f^0_{145} + 2f^0_{445}), \\
f_{555} &= f^0_{555} + 3y(3f^0_{155} + f^0_{455} + f^0_{555}) + 3y_{12}(3f^0_{115} + f^0_{445} + f^0_{555} + 6f^0_{125} + 6f^0_{145} + 6f^0_{155} + 2f^0_{455}) + y_{123}(3f^0_{111} + \\
&\quad f^0_{444} + f^0_{555} + 18f^0_{112} + 9f^0_{114} + 9f^0_{115} + 6f^0_{123} + 18f^0_{124} + 18f^0_{125} + 9f^0_{144} + 9f^0_{155} + 18f^0_{145} + 3f^0_{445} + 3f^0_{455}).
\end{aligned} \tag{44}$$

So from (the generalisation of) Eq. (32) we know how to express the c_{abc} in terms of the b and y. Given that we have isotropic Potts models with the same coupling constants, the non-trivial selfduality relations read simply:

$$c_{444} = c_{555}, \qquad c_{455} = c_{445}, \qquad c_{155} = c_{144}, \qquad c_{115} = c_{114}, \qquad c_{125} = c_{124}. \tag{45}$$

A. Selfdual solutions

There are two types of trivial solutions, as in the case of two coupled models.

An important difference with the case of two coupled models is that non-trivial selfdual solutions with $y = y_{12} = y_{123} = 0$ only exist for exceptional values of q (i.e., $q = 0$ and $q = 4$; see below). So for generic values of q, the three-spin interactions are necessary to generate non-trivial solutions of the selfduality problem.

As before, the most general solutions are not algebraically simple in terms of the variables b and y. We therefore report only a few special cases:

1. There are three non-trivial solutions with $L_{12} = L_{123} = 0$ (i.e., $y_{12} = y_1^2$ and $y_{123} = y_1^3$). They read:

$$b_1 = -\frac{q}{2}, \qquad b_{12} = \frac{q^2}{4}, \qquad b_{123} = \frac{q^3 - 9q^2 + 18q - 12}{4}, \qquad y_1 = \frac{q(4-q)}{q^2 - 4q + 4}\,; \tag{46}$$

$$b_1 = -1, \qquad b_{12} = \frac{q}{2}, \qquad b_{123} = \frac{q(1-q)}{2}, \qquad y_1 = \frac{q}{2-q}\,; \tag{47}$$

$$\begin{aligned} &(4q-6)b_1^2 + 2qb_1 + q = 0, \qquad b_{12} = -\frac{q(1+2b_1)}{2}, \\ &b_{123} = \frac{q^2((8q^2-16q-6)b_1 + 4q^2 - 12q + 3)}{4(3b_1+q)(2q-3)}, \qquad y_1 = \frac{q}{2-q}. \end{aligned} \tag{48}$$

2. There are two non-trivial solutions with $L_1 = 0$ (i.e., $y_1 = 0$):

$$b_1 = -1, \qquad b_{12} = \frac{q}{2}, \qquad b_{123} = \frac{q(1-q)}{2}, \qquad y_{12} = \frac{q(4-q)}{q^2-4q+4}, \qquad y_{123} = \frac{2q(q^2-6q+6)}{q^3-6q^2+12q-8}\,; \tag{49}$$

$$\begin{aligned} &b_1 = 1-q, \qquad b_{12} = \frac{q(1-q)}{2-q}, \qquad b_{123} = \frac{q(q^2-3q+1)}{(2-q)(q-3)}, \\ &y_{12} = \frac{q(q^6 - 11q^5 + 45q^4 - 87q^3 + 86q^2 - 42q + 8)}{q^6 - 18q^5 + 126q^4 - 432q^3 + 756q^2 - 648q + 216}, \\ &y_{123} = \frac{q(4q^8 - 72q^7 + 531q^6 - 2068q^5 + 4584q^4 - 5856q^3 + 4220q^2 - 1584q + 240)}{q^9 - 30q^8 + 390q^7 - 2872q^6 + 13140q^5 - 38520q^4 + 71928q^3 - 82080q^2 + 51840q - 13824}. \end{aligned} \tag{50}$$

3. There are two non-trivial solutions with $y_{12} = 0$. We give here only the first one because the other is complicated:

$$b_1 = -1, \qquad b_{12} = \frac{q}{2}, \qquad b_{123} = \frac{q(1-q)}{2}, \qquad y_1 = \frac{q(4-q)}{2(q^2-4q+4)}, \qquad y_{123} = \frac{q^2(q-6)}{2(q^3-6q^2+12q-8)}. \tag{51}$$

Note that for $q = 2$ the non-trivial solutions given are singular. In fact, for $q = 2$, they can be written as: $b_1 = -1$, $b_{12} = 1$, $b_{123} = -1$, and the values of y, y_{12}, y_{123} are arbitrary. Indeed the values of the b correspond to three decoupled antiferromagnetic Potts models at zero temperature, and so the values of the parameters y_1, y_{12}, y_{123} do not matter.

V. NUMERICAL STUDY OF TWO COUPLED MODELS

It was mentioned in the Introduction that the Potts model is usually critical on the selfdual manifolds, for suitable values of q (i.e., $0 \le q \le 4$). We expect this also to be true for coupled Potts models, and so it is interesting to determine the corresponding universality classes. In the lack of an exact (Bethe Ansatz) solution, this question can be addressed by evaluating the effective central charge along the selfdual manifolds, e.g., using numerical transfer matrix techniques.

In this Section we focus on the selfdual curve (19) for two coupled models with pure two-spin interactions. Note that in Section II D we have already remarked on a few special values of the parameter g for which the physics of the two coupled models can be related to that of a single model.

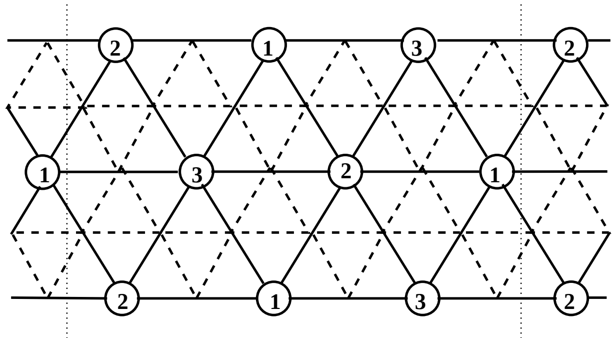

FIG. 5: Semi-infinite strip, here of size $L = 2$ triangles in the finite (vertical) direction. Periodic boundary conditions identify the top and the bottom of the figure. Potts spins are defined at the loci of the small circles; they interact along the solid lines which form a triangular lattice. The labels within each circle identify the usual three sublattices of the triangular lattice. The loop model is defined on the medial Kagomé lattice, shown in broken linestyle. The transfer matrix propagates the system along the horizontal direction, from left to right. Thin dotted lines indicate successive time slices (see text).

A. Transfer matrix

The triangular-lattice Potts model can be transformed into a loop model on the medial (surrounding) graph—which is the Kagomé lattice—in a standard way [3, 11]. (This loop model should not be confused with the one described in Section III.) We have computed the free energy of the two coupled models (6) on semi-infinite strips by constructing the transfer matrix of two coupled Kagomé-lattice loop models.

The geometry is depicted in Fig. 5. For a periodic strip of circumference L triangles, each time slice cuts $2L$ dangling ends of the Kagomé-lattice loop model. In order to have the leading eigenvalue Λ_0 of the transfer matrix $\mathbf{T}_L$ correspond to the ground state of the continuum model, the definition of $\mathbf{T}_L$ must respect the usual sublattice structure of the triangular lattice. This means that L must be even, and that successive time slices are as shown in Fig. 5.

The numerical diagonalisation is most efficiently performed by decomposing $\mathbf{T}_L$ in a product of sparse matrices, each adding one vertex of the Kagomé lattice (or, equivalently, one edge of the triangular lattice). With the setup of Fig. 5, all these sparse matrices are identical, except for the position of the two dangling ends on which they act. We have been able to diagonalise $\mathbf{T}_L$ for sizes up to $L = 10$ (the corresponding matrix has dimension 141 061 206).

B. Central charge

The free energy per unit area is $f(L) = -\frac{1}{4\sqrt{3}L} \log \Lambda_0$, the length scale being the height of one triangle. We have extracted values of the effective central charge c from three-point fits of the form [17]

$$f(L) = f(\infty) - \frac{\pi c}{6L^2} + \frac{A}{L^4}, \tag{52}$$

where the non-universal term in A is supposed to adequately represent the higher-order corrections.

Fig. 6 shows the numerical values of $c(g)$ along the curve (19). For each value of g, three estimates for $c(g)$ are shown, obtained by fitting $\{f(L-4), f(L-2), f(L)\}$ to (52) for $L = 6$, $L = 8$ and $L = 10$ respectively.

Naively, one would expect the K_{12} coupling to be marginal at the $q = 2$ ferromagnetic point (i.e., at $g = \frac{3}{4}$) and the surrounding regime to be accessible to perturbative calculations. However, it should be remembered that 1) the point $g = \frac{3}{4}$ is not that of two decoupled Ising models but that of a single 4-state model, and that 2) the renormalisation group equations for N coupled models are singular when $N = 2$ [11]. Nevertheless, the numerics seems quite conclusive that the $q < 2$ regime with $\frac{1}{4} \leq g < \frac{3}{4}$ has a central charge which is just twice that of (21) [upon changing the parametrisation, $g \to 1 - g$]:

$$c(g) = 2\left(1 - \frac{6(1-g)^2}{g}\right), \qquad \text{for } \frac{1}{4} \leq g < \frac{3}{4}, \tag{53}$$

meaning presumably that the continuum limit is really that of two decoupled models. This is also consistent with

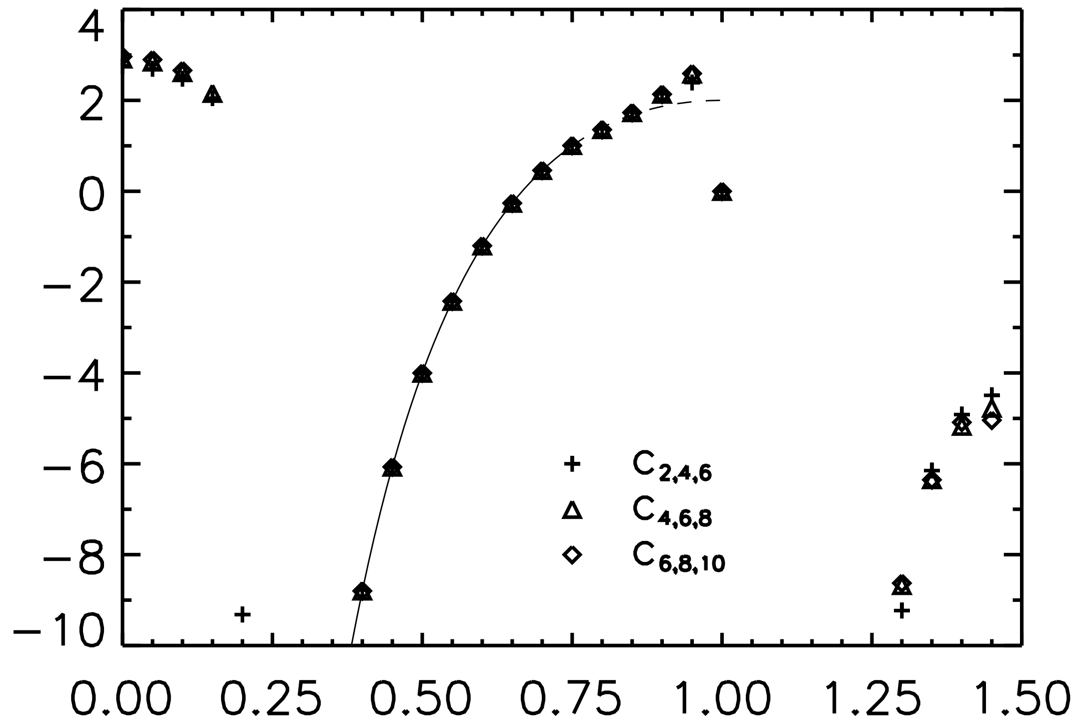

FIG. 6: Three-point fits for the effective central charge c as a function of g. The solid curve shows the exact result (53), valid for $\frac{1}{4} \leq g < \frac{3}{4}$, as discussed in the text.

the result $c(g = \frac{1}{2}) = -4$. The agreement of the numerics with (53) is excellent also for those data points (with $0.25 \leq g \leq 0.35$) which are not visible in Fig. 6.

The region $0 \leq g \leq g_1$, with $g_1 \approx 0.15$, is interesting as it corresponds to $c > 2$. This hints at the coupled models requiring some kind of higher symmetry than its two constituent bosonic theories. Note in particular that for $g = 0$, our three estimates for c read $c_{2,4,6} = 2.758$, $c_{4,6,8} = 2.914$ and $c_{6,8,10} = 2.966$, which we extrapolate to $c(g = 0) = 3.00 \pm 0.01$. We conjecture that the exact result is $c(g = 0) = 3$. Since $q = 4$, this theory can also be represented as two coupled vertex models on the Kagomé lattice [3].

For $g = 1$, the Boltzmann weights (20) are all ± 1. It turns out that in this particular case it is more convenient to work with a modified transfer matrix that adds not one but $L/2$ time slices, cf. Fig. 5. This matrix has its largest eigenvalue equal to unity regardless of L, and we conclude that $f(L) = 0$ for any L. In particular, this means $c(g = 1) = 0$. The numerics is however indicative of a non-trivial regime for $\frac{3}{4} < g < 1$, and it seems that we may have $c(g) \to 4$ as $g \to 1^-$, consistent with two decoupled models each of which is obtained by taking the limit $g \to 1^+$ in (21).

In the region $1 < g < g_2$, with $g_2 \approx 1.10$, our numerical diagonalisation scheme experiences difficulties, maybe due to the leading eigenvalue having a non-zero imaginary part.

Finally, in the regime $g_2 < g < \frac{3}{2}$ the central charge takes large, negative values (in particular, some of the values are not visible in Fig. 6). At first sight one might believe that the continuum limit is that of two decoupled models in the Berker-Kadanoff phase, i.e., with $c(g)$ given by twice that of (21) [upon changing the parametrisation $g \to 2 - g$]: $c(g) = 2\big(1 - 6(2-g)^2/(g-1)\big)$. This possibility is however clearly ruled out by the numerics, and $c(g)$ appears to be given by a non-trivial expression.

As $g \to \frac{3}{2}$, the two leading eigenvalues of the transfer matrix become degenerate. In the sense of analytically continuing the curve (18) to negative values of q, this presumably marks a transition to non-critical behaviour for $q < 0$, i.e., with the phase transitions being first-order in b upon crossing the curve (18).

VI. CONCLUSION

Using a mapping of coupled Potts models on the triangular lattice to coupled loop models, we have obtained non-trivial selfdual manifolds for two and three coupled Potts models with two and three-spin interactions. A numerical study of the case of two coupled models shows that these manifolds are good candidates for novel critical points, in particular in the antiferromagnetic and unphysical regimes.

The technique can be applied to any number of coupled models, but expressing the solutions explicitly in terms of the original coupling constants becomes increasingly complicated as the number of models grows. This is in contrast to the quite simple results for coupled Potts models on the square lattice [12, 13].

It would be interesting to study the simplest non-trivial case (19) using the methods of integrable systems.

Acknowledgments

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